

Photons and static gravity

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Abstract

The influence of static gravitational field on frequency, wave-length and velocity of photons and on the energy levels of atoms and nuclei is considered in the most elementary way. The interconnection between these phenomena is stressed.

1 Introduction

The behaviour of light in a static gravitational field has a long history. It has been discussed in many monographs on General Relativity and in vast popular literature. (An extensive list of references may be found in the recent publications [1, 2].)

In this note we discuss from a single point of view four effects which in the literature are usually discussed separately:

- 1) the gravitational redshift of photons flying away from a gravitating body,
- 2) the increase of energy difference between levels in atoms with the increase of their distance from gravitating body,
- 3) the retardation of radar echo from planets,
- 4) the deflection of star light observed during a solar eclipse.

The relation between the effects 1) and 2) was thoroughly discussed in refs. [1, 2]. This note may be considered as a continuation of those articles.

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2 Redshift and clocks

The phenomenon of gravitational redshift is widely explained by using (explicitly or implicitly) a presumed analogy between a photon and a stone: when moving away from a gravitating body (sun or earth) both a photon and a stone are supposed to lose energy overcoming the gravitational attraction.

This explanation, as was stressed recently [1, 2], is wrong. The energy of a photon E and hence its frequency $\omega = E/\hbar$ do not depend on the distance from the gravitating body, because in the static case the gravitational potential does not depend on the time coordinate t . The reader who is not satisfied with this argument may look at Maxwell's equations as given e.g. in section 5.2 of ref. [3]. These equations with time independent metric have solutions with frequencies equal to those of the emitter.

In what follows we will often use both the quantum notion of photon and the classical notions of electromagnetic wave or even ray of light. To a certain extent this is caused by the physical nature of experiments in which the effects we are discussing were observed. For instance, in the famous experiments by Pound et al. the γ -quanta were flying vertically in a tower. The language of quantum physics is necessary to discuss the behavior of nuclear and atomic levels, and hence of atomic clocks. On the other hand in the discussion of the radar echo and deflection of light quantum effects are absolutely not essential.

The proper explanation of gravitational redshift lies in the behavior of clocks (atoms, nuclei). The rest energy E_0 of any massive object increases with increase of the distance from the gravitating body because of the increase of the potential ϕ :

$$E_0 = mc^2(1 + \phi/c^2) . \quad (1)$$

The gravitational potential is

$$\phi = -GM/r , \quad (2)$$

where $G = 6.67 \cdot 10^{-8} \text{ cm}^3/\text{g} \cdot \text{s}^2$ is the Newton constant, M is the mass of the body, and r – the distance from its center; $M = 2 \cdot 10^{33} \text{ g}$ for the sun, while for the earth $M = 6 \cdot 10^{27} \text{ g}$.

The universal increase of rest energy of an atom in an excited and the ground state means that the energy difference between levels also universally

increases by the factor $(1 + \phi/c^2)$. As a result of this increase the energy of a photon emitted in a transition of an atom downstairs is not enough to excite a reverse transition upstairs. For the observer upstairs this looks like a redshift of the photon.

A “semi-competent” observer who knows nothing about the dependence of the rate of the standard clocks (atoms, nuclei) on the value of gravitational potential could conclude that photon is redshifted not relatively, but absolutely. A competent observer knows about special experiments on airplanes which proved that, in accord with General Relativity, (atomic) clocks run faster high above the earth (see e.g. refs. [1, 2]). Therefore for a competent observer the apparent redshift of the photon is a result of the blueshift of the clock.

A naive (but obviously wrong!) way to derive the formula for the redshift is to ascribe to the photon with energy E a mass $m_\gamma = E/c^2$ and to apply to the photon a non-relativistic formula $\Delta E = -m_\gamma \Delta\phi$ treating it like a stone. Then the relative shift of photon energy is $\Delta E/E = -\Delta\phi/c^2$, which coincides with the correct result. But this coincidence cannot justify the absolutely thoughtless application of a nonrelativistic formula to an ultrarelativistic object.

It is interesting that one can still speak about the redshift of the photon if one considers not its frequency, but its wavelength or momentum¹. To see this let us consider the condition that photon is massless:

$$g^{ij}p_i p_j = 0 \quad , \quad (3)$$

where g^{ij} , $i, j = 0, 1, 2, 3$ are contravariant components of the metric tensor. In a spherically symmetric potential when photon moves along a radius we have

$$g^{00}p_0 p_0 - g^{rr}p_r p_r = 0 \quad . \quad (4)$$

In the case of standard Schwarzschild metric [3, 4, 5]:

$$g^{00} = (1 - \frac{r_g}{r})^{-1} \quad , \quad (5)$$

$$g^{rr} = (1 - \frac{r_g}{r}) \quad , \quad (6)$$

¹It is instructive to compare the behavior of λ in the spherically symmetric case with that in a linear potential near the earth’s surface considered in detail in refs. [1, 2].

where

$$r_g = 2MG/c^2 . \quad (7)$$

For the sun $r_g = 3$ km.

We identify the covariant vector p_i (and not contravariant vector p^i) with energy - momentum four-vector because

$$p_i = \partial S / \partial x^i , \quad (8)$$

where S is action. This guarantees that if time is uniform (g^{ij} is time independent), then energy of a particle is conserved.

As the energy of the photon $E = cp_0$ does not depend on r , we immediately see from Eq.(4) that its momentum p_r does depend:

$$p_r = \frac{E}{c} \left(1 - \frac{r_g}{r}\right)^{-1} . \quad (9)$$

The closer the photon to the sun, the larger its momentum. Hence its wavelength is blueshifted:

$$\lambda = \frac{2\pi\hbar}{p_r} = \frac{2\pi\hbar c}{E} \left(1 - \frac{r_g}{r}\right) \quad (10)$$

In that sense (but only in that!) the widely used words about redshift and blueshift of the photon are correct.

3 Photon velocity

With ω being constant and λ decreasing with radius the velocity of the photon

$$v = \frac{\lambda\omega}{2\pi} = c \left(1 - \frac{r_g}{r}\right) \quad (11)$$

also decreases with decreasing radius r . In that respect photon drastically differs from a non-relativistic stone, falling into the sun. (The velocity of the stone obviously increases with the decrease of r .)

The same conclusion could be reached by an explicit calculation of velocity from the expression for the interval:

$$ds^2 = g_{00}c^2dt^2 - g_{rr}dr^2 = 0 , \quad (12)$$

where

$$g_{00} = 1/g^{00} = (1 - r_g/r) \quad (13)$$

$$g_{rr} = 1/g^{rr} = (1 - r_g/r)^{-1} \quad (14)$$

$$v = dr/dt = c \left(\frac{g_{00}}{g_{rr}} \right)^{1/2} = c(1 - r_g/r) \quad (15)$$

[Note that if we defined (incorrectly!) v as in Special Relativity, $v = pc^2/E = cp_r/p_0$, we would get a different behaviour of velocity: $v = c(1 - r_g/r)^{-1}$, because momentum increases both for the photon and for the stone.]

The time t is often called the coordinate time, or world time, as it is set by clocks infinitely far from the gravitating body. Therefore it is proper to call v the coordinate or world velocity. It is worth noting that while the velocity v changes with the potential, the velocity of light in a locally inertial reference frame is always equal to c .

The decrease of coordinate velocity v discussed above leads to two types of effects both of which are well known. One of them is the famous deflection of light by the sun, predicted by Einstein and first observed during 1919 solar eclipse. This observation was reported by newspapers worldwide and started the cult of Einstein. The other effect is not so widely known. It is the delay of radar echo from inner planets (Venus, Mercury), the measurement of which was proposed [6] and realized by I.I.Shapiro. (For the description of the measurements see ref. [3], Section 8.7.)

It is appropriate to discuss both effects (the echo and the deflection) in the framework of geometrical optics (see ref. [5]), as the wave-length λ is negligible compared to any other characteristic length. The refraction index n is given by:

$$n = \frac{c}{v} = (1 - r_g/r)^{-1} = (1 + r_g/r) \quad (16)$$

Here and in what follows we consistently neglect terms of the order of $(r_g/r)^2$: the gravitational field in the solar system is weak, and therefore

$$n - 1 = r_g/r \ll 1 \quad (17)$$

4 Radar echo

Let us start with radar echo and consider for simplicity the echo from the sun. (This is of course only a gedanken experiment). The duration of a two

way travel of a signal in this case is

$$t = 2 \int_{R_s}^{r_e} \frac{dr}{v(r)} = \frac{2}{c} \int_{R_s}^{r_e} n(r) dr = \frac{2}{c} \int_{R_s}^{r_e} (1 + r_g/r) dr \quad (18)$$

and the delay compared to the case $v = c$ is

$$\Delta t = 2 \frac{r_g}{c} \ln \frac{r_e}{R_s} , \quad (19)$$

where $r_e = 150 \cdot 10^6$ km is the distance between the sun and the earth, while $R_s = 0.7 \cdot 10^6$ km is the radius of the sun.

Consider now the echo from a planet close to its superior conjunction (when the planet is at the largest distance from the earth, so that the radar ray almost grazes the sun). In that case, unlike the previous ones, we have a non-radial trajectory with a non vanishing impact parameter ρ . Let us denote by z the coordinate along the straight line connecting the earth at (z_e, ρ) and the planet at (z_p, ρ) . (The axis ρ is obviously orthogonal to z .) Note that we can use straight lines even for non-radial trajectories because the gravitational field in solar system is weak: $(n - 1) = r_g/r \ll 1$. The deflection of light (which is considered in the next section) gives a negligible contribution of order $(n - 1)^2$ to the echo delay time. To calculate this delay time we use isotropic metric:

$$\begin{aligned} \Delta t &= \frac{2}{c} \int_{z_e}^{z_p} dz \left(\frac{c}{v} - 1 \right) = \frac{2}{c} \int_{z_e}^{z_p} dz (r_g/r) = \\ &= 2 \frac{r_g}{c} \int_{z_e}^{z_p} \frac{dz}{\sqrt{z^2 + \rho^2}} = 2 \frac{r_g}{c} \left(\ln \frac{|z_p| + r_p}{\rho} + \ln \frac{|z_e| + r_e}{\rho} \right) = \\ &= 2 \frac{r_g}{c} \ln \frac{4r_p r_e}{\rho^2} . \end{aligned} \quad (20)$$

The effect is maximal when $\rho \simeq R_s$. The logarithmic dependence for the echo from the sun and from a planet is the same. The non-logarithmic terms ($\sim r_g/r_e$) are neglected here: we do not take into account the retardation of clocks on the earth with respect to the world time. Their inclusion would increase Δt by $\sim 9\%$. For Mercury $r_p = 58 \cdot 10^6$ km, hence $\Delta t \simeq 240 \mu s$.

It is interesting to note that retardation must take place not only for photons but also for any extreme relativistic particle if its mass m and energy E are such that

$$(mc^2)^2/2E^2 \ll r_g/r_e \quad (21)$$

For the sun - earth system $r_g/r_e \sim 2 \cdot 10^{-8}$, while for electrons from LEP I $(mc^2/E)^2 \sim 10^{-10}$. It is easy to see that even for low energy solar and reactor neutrinos the inequality is also fulfilled. Unfortunately echo experiments with electrons and neutrinos are not realistic for many reasons.

5 Deflection of light

Consider a light ray connecting a star and the earth. In this case the trajectory of light is bent, but we will continue to denote the variable along this trajectory as z (its end points z_* and z_e , respectively) and the impact parameter as ρ . Assume that we deal with a “thick pencil” ray of diameter $d\rho$. It is easy to see from Eq. (15) that the inward side of the ray moves slower than the outward one. (Here inward means closer to the sun.) As a result the cross-section of the ray is tilted: when its inward edge propagates by dz the outer edge propagates by $dz[1 + (\partial v/c\partial\rho)d\rho]$. Thus the angle of the tilt is

$$d\alpha = -dz(\partial v/c\partial\rho) = \frac{dz\partial(r_g/r)}{\partial\rho} . \quad (22)$$

Hence the direction of the ray would be inward deflected, the differential deflection angle of the ray being given by Eq. (22), while the total deflection angle

$$\alpha = \int_{z_*}^{z_e} dz \frac{\partial(r_g/r)}{\partial\rho} = \frac{\partial}{\partial\rho} \int_{z_*}^{z_e} dz (r_g/r) . \quad (23)$$

If we now substitute z_* by z_p which is not essential, because both of them are much larger than ρ and will not enter the result, we get from Eq. (20):

$$\alpha = \frac{\partial}{\partial\rho} \left(\frac{c\Delta t}{2} \right) = -\frac{2r_g}{\rho} . \quad (24)$$

As was stressed in ref. [5], the first part of Eq. (24) is a consequence not of General Relativity, but of geometrical optics.

For the case of solar eclipse $\rho = R_s$ and

$$\alpha = -2r_g/R_s = -1.75'' . \quad (25)$$

This is exactly the angle predicted by Einstein [7] in 1915 in the framework of General Relativity. He used the coordinate velocity v given by Eq.(15). Note that in 1911 by implicitly assuming that $g_{rr} = 1$ and, hence, using $v = c(1 - r_g/2r)$ Einstein derived an expression for the angle in which coefficient 2 was missing [8].

6 Conclusions

Thus in a static gravitational field the frequency ω of the photon is a constant; it is equal to the frequency at emission. The phenomenon of gravitational redshift is explained by the increase of the energy difference between levels in atoms or nuclei (in general, by the increase of the rate of clocks) with the increase of their distance from the gravitating body. The constant energy of photon appears as redshifted with respect to the blueshifted energy difference of atomic levels.

The momentum of a photon p and hence its wave-length λ change with the decrease of the distance from gravitating body: p increases, λ decreases. Hence the coordinate (world) velocity v of the photon decreases. This explains the two well established effects: the delay of radar echo from planets and the deflection of star light by the sun.

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